

POINTS AND LINES PROBLEMS

- Line equation by two points
- Relative location of lines and points
- Segments intersection
- Left and right turns of two line segments

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LINE EQUATION BY TWO POINTS

Let the coordinates of points P_1 and P_2 be equal to (x_1, y_1, w_1) and (x_2, y_2, w_2) , respectively. A point with coordinates (x, y, w) will be collinear with points P_1 and P_2 if its coordinates are linearly dependent on the coordinates of these points. This means that the determinant of the matrix, the columns which specify the coordinates of the three points under consideration, must be equal to 0.

$\det \begin{bmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ w & w_1 & w_2 \end{bmatrix} = 0.$ This equation defines a straight line passing through two points P_1 and P_2 . Having opened the determinant, we get the equation:

$$x(y_1 w_2 - w_1 y_2) + y(w_1 x_2 - x_1 w_2) + w(x_1 y_2 - y_1 x_2) = 0.$$

LINE EQUATION BY TWO POINTS

The standard line equation is $Ax + By + Cw = 0$. Use the previous formula we can determine each line coefficient:

$$A = (y_1w_2 - w_1y_2)$$

$$B = (w_1x_2 - x_1w_2)$$

$$C = (x_1y_2 - y_1x_2).$$

Convert from homogenous coordinates to Euclidian ($w=1$) we have:

$$A = (y_1 - y_2)$$

$$B = (x_2 - x_1)$$

$$C = (x_1y_2 - y_1x_2).$$

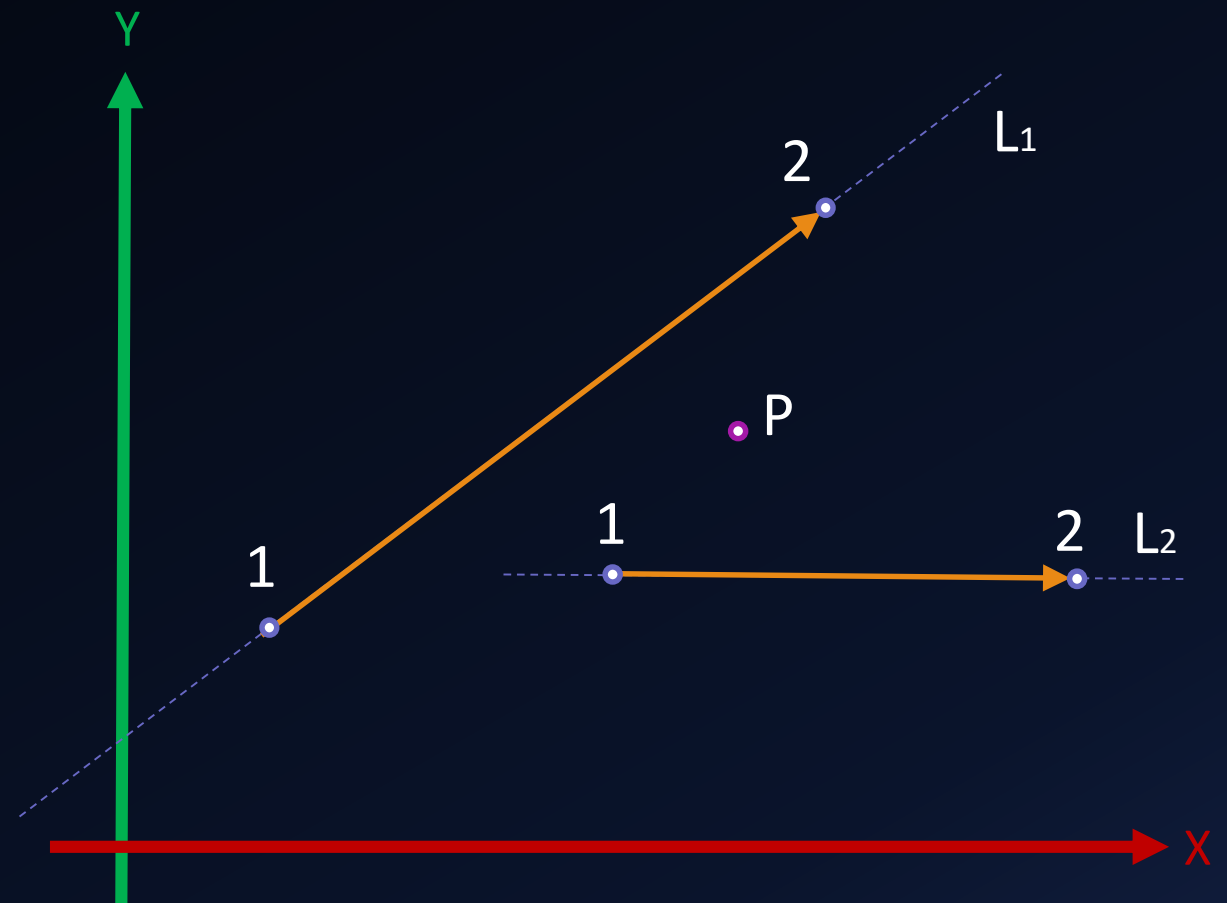
For two parallel lines we have condition:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2}$$

RELATIVE LOCATION OF LINES AND POINTS

Solving many problems related to determining the visibility of image objects and clipping problems requires determining the relative location of lines and points.

Agreement. In the absence of special notes, all straight-line segments are oriented: for non-horizontal segments, the arrows point from bottom to top, and for horizontal segments — from left to right.



RELATIVE LOCATION OF LINES AND POINTS

Definition. We will assume that point **P** is located to the right of line **L** if it is placed to the right of the observer moving along this line from the first endpoint of the segment to the second one.

Assertion. Let **X**, **Y**, and **W** be the coordinates of point **P**, and let (x_1, y_1, w_1) and (x_2, y_2, w_2) be the endpoints of the line **L** segment. If the values of **W**, w_1 , and w_2 are positive, then point **P** is to the right of the line, determined by the line **L** segment, if and only if the inequality holds:

$$X(y_1w_2 - w_1y_2) + Y(w_1x_2 - x_1w_2) + W(x_1y_2 - y_1x_2) < 0.$$

Passing from homogeneous coordinates to Euclidean statements remains true if we accept $W=w_1=w_2=1$.

$$X(y_1 - y_2) + Y(x_2 - x_1) + (x_1y_2 - y_1x_2) < 0.$$

SEGMENTS INTERSECTION

If two line segments are given by points P_1 , P_2 , P_3 , and P_4 , then they intersect if and only if, when substituting the coordinates of points P_1 and P_2 into the equation of the line connecting the points P_3 and P_4 , the results have different signs. A similar condition applies when points P_1 , P_2 and P_3 , P_4 exchange roles. In fact, we need to determine the signs of the following four values:

$$S_1 = x_1(y_3w_4 - w_3y_4) + y_1(w_3x_4 - x_3w_4) + w_1(x_3y_4 - y_3x_4);$$

$$S_2 = x_2(y_3w_4 - w_3y_4) + y_2(w_3x_4 - x_3w_4) + w_2(x_3y_4 - y_3x_4);$$

$$S_3 = x_3(y_1w_2 - w_1y_2) + y_3(w_1x_2 - x_1w_2) + w_3(x_1y_2 - y_1x_2);$$

$$S_4 = x_4(y_1w_2 - w_1y_2) + y_4(w_1x_2 - x_1w_2) + w_4(x_1y_2 - y_1x_2).$$

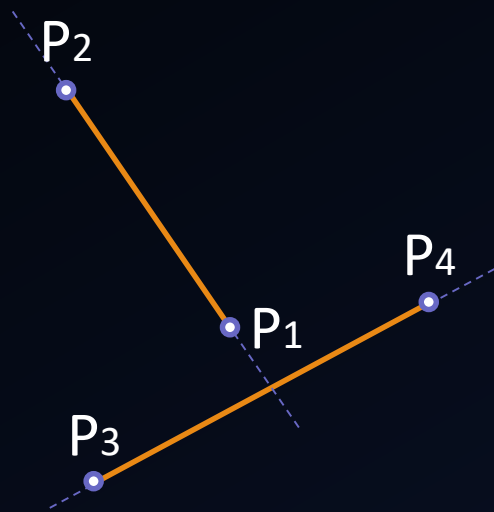
The intersection condition requires that S_1 and S_2 have different signs, and the same applies to S_3 and S_4 .

SEGMENTS INTERSECTION

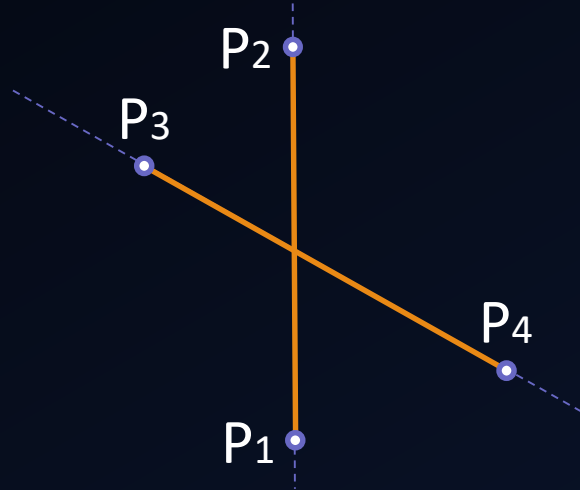
If the conditions are met, then the coordinates of the intersection point can be determined by solving a pair of linear equations:

$$x(y_1w_2 - w_1y_2) + y(w_1x_2 - x_1w_2) + w(x_1y_2 - y_1x_2) = 0$$

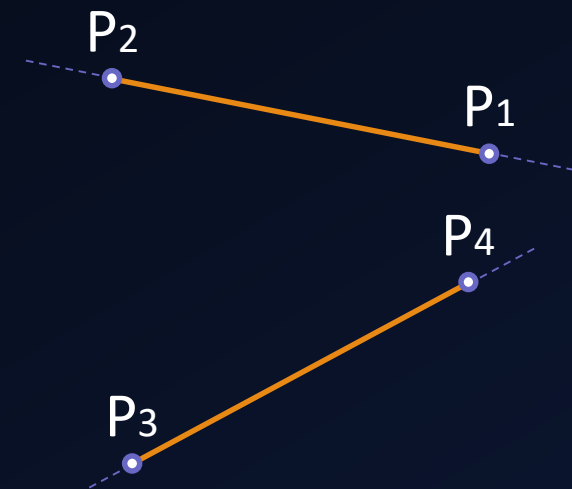
$$x(y_3w_4 - w_3y_4) + y(w_3x_4 - x_3w_4) + w(x_3y_4 - y_3x_4) = 0$$



$$S_1S_2 > 0 \text{ and } S_3S_4 < 0$$



$$S_1S_2 < 0 \text{ and } S_3S_4 < 0$$



$$S_1S_2 > 0 \text{ and } S_3S_4 > 0$$

SEGMENTS INTERSECTION

The intersection condition can be written in a compact form:

$$S_1 = \det(P_1, P_3, P_4); S_2 = \det(P_2, P_3, P_4); S_1 S_2 < 0$$

$$S_3 = \det(P_3, P_1, P_2); S_4 = \det(P_4, P_1, P_2); S_3 S_4 < 0$$

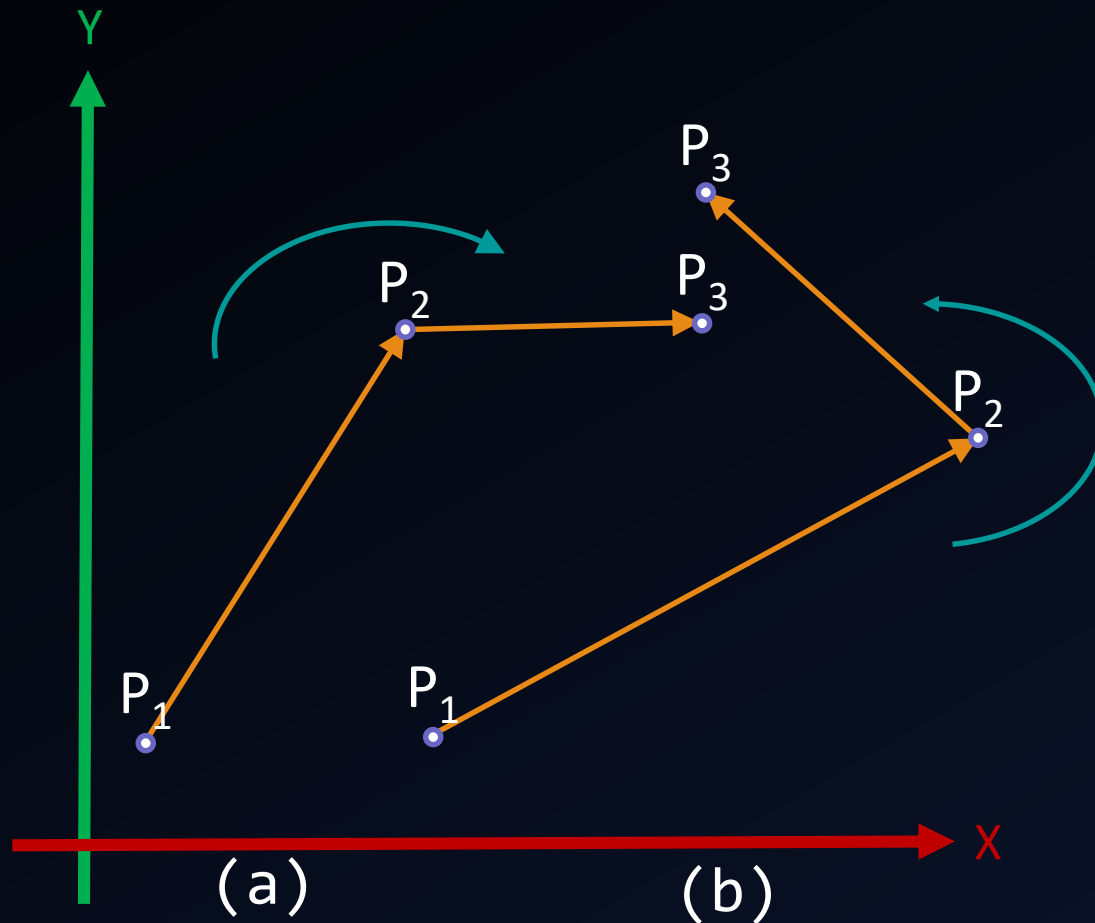
The pair of linear equations to find the intersection point will be:

$$\det(P, P_1, P_2) = 0; \det(P, P_3, P_4) = 0$$

If one or more values of S_i have zero values, then a degenerate case occurs. For example, if $S_1=0$, it means that point P_1 is on a straight line between points P_3 and P_4 , in case $S_3 S_4 < 0$, or on the same straight line but above (left) or below (right) segment $P_3 P_4$. Where exactly the point of intersection lies depends on the signs of the values S_3 and S_4 .

LEFT AND RIGHT TURNS OF TWO LINE SEGMENTS

To get the intuition of what is left and what is right turn, consider an example shown below.

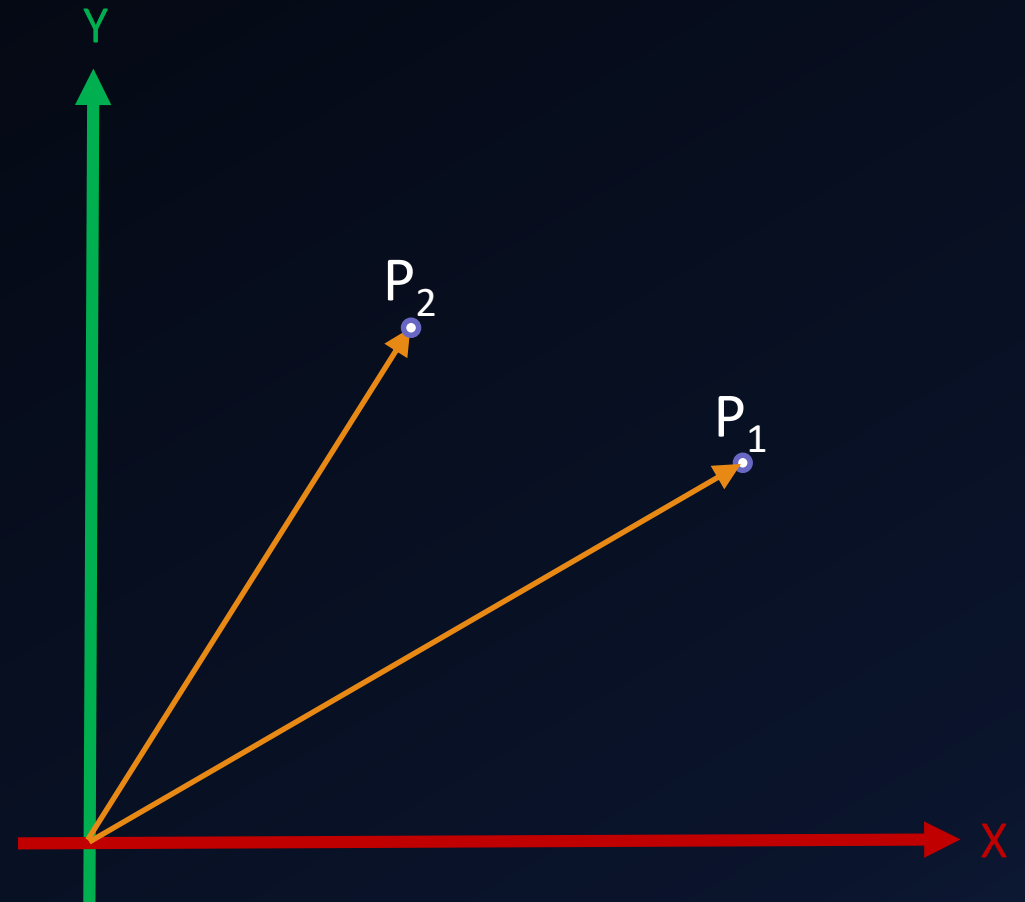


In both figures, there are three points P_1 , P_2 , and P_3 , and two line segments P_1P_2 and P_2P_3 . Point P_2 is common to both line segments. In Figure (a), segment P_2P_3 is making a right turn at point P_2 ; in Figure (b), segment P_2P_3 is making a left turn at the common point P_2 . It is easier for our eyes to quickly identify whether a segment is making a left turn or a right turn just by looking at the figure because your eyes can identify things very quickly. Next, we consider how computers identify this using geometric algorithms.

LEFT AND RIGHT TURNS OF TWO LINE SEGMENTS

Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, we need to first determine whether point P_1 is clockwise or counter-clockwise from point P_2 with respect to the origin. There is an easier and more efficient solution to this than finding the angle which is calculating the cross product of the vectors P_1 and P_2 . Mathematically the cross product of two vectors P_1 and P_2 is given by $P_1 \times P_2 = x_1 * y_2 - x_2 * y_1$. If the value of $P_1 \times P_2$ is positive then P_1 is clockwise from P_2 with respect to origin as shown in the figure right.

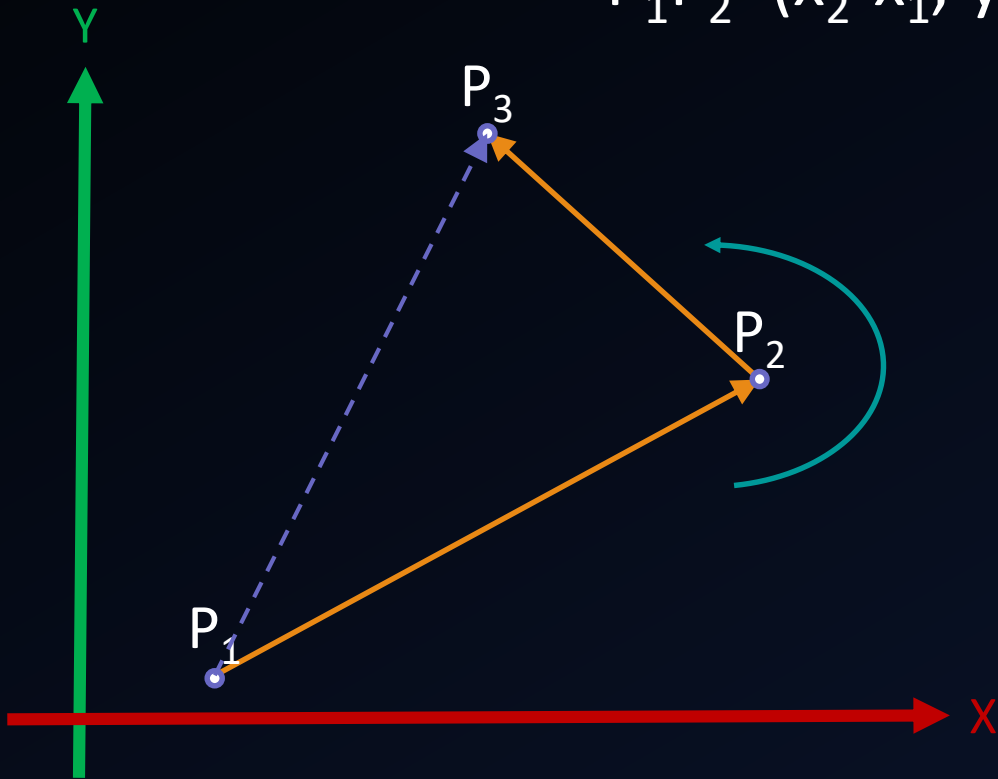
Similarly, if $P_1 \times P_2$ is negative then P_1 is counter-clockwise from P_2 with respect to the origin if the value is 0 then points P_1 , P_2 , and origin are collinear.



LEFT AND RIGHT TURNS OF TWO LINE SEGMENTS

Consider two segments whose end points are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_1(x_1, y_1)$, $P_3(x_3, y_3)$ respectively. In order to calculate the cross-product of two segments, we need to convert them into vectors. This can be done in the following way:

$$\vec{P_1P_2} = (x_2 - x_1, y_2 - y_1), \vec{P_1P_3} = (x_3 - x_1, y_3 - y_1)$$



To determine if a segment P_2P_3 turns left from a segment P_1P_2 at point P_2 , we draw the next vector P_1P_3 and check if the new vector is counter-clockwise from a vector P_1P_2 . In the figure left, the P_1P_3 vector is counter-clockwise from vector P_1P_2 , so it turns left. If P_1P_3 vector is clockwise from vector P_1P_2 then P_2P_3 turns right from vector P_1P_2 at the point P_2 .



Thank you!